Name of college - S.S. college
J-Bad

Deptt - Mathematics
Topic - Problem based on ellipse
(Analytical Geometry of 2-D)

Date - 15-07-2020
Time - 12.45 p.m to 1.30 P.M
Class - B.Sc I (Sub)

By Dr- Shree Nath Sharma

1. Find the Centre, the length of the axes and the eccentricity of the ellipse
$$2x^2+3y^2-4x-12y+13=0$$

Solution The Given Equation is
$$2x^2+3y^2-4x-12y+13=0$$
$$\Rightarrow 2x^2-4x+3y^2-12y+13=0$$
$$\Rightarrow 2[x^2-2x+1]+3[y^2-4y+4]-14+13=0$$
$$\Rightarrow 2(x-1)^2+3(y-2)^2=1$$
$$\Rightarrow \frac{(x-1)^2}{1/2}+\frac{(y-2)^2}{1/3}=1$$

put $x-1=X$ $y-2=Y$
$$\Rightarrow \frac{X^2}{1/2}+\frac{Y^2}{1/3}=1$$

Centre $X=0 \Rightarrow x-1=0 \Rightarrow x=1$
$Y=0 \Rightarrow y-2=0 \Rightarrow y=2$.

$a^2=1/2$ $b^2=1/3$
$a=\frac{1}{\sqrt2}$ $b=\frac{1}{\sqrt3}$

~~$x=a$ y~~

Major axis $=2a=2\times\frac{1}{\sqrt2}=\sqrt2$
Minor axis $=2b=2\times\frac{1}{\sqrt3}=\frac{2}{\sqrt3}$

$b^2=a^2(1-e^2)$
$\frac{1}{3}=\frac{1}{2}(1-e^2) \Rightarrow 1-e^2=2/3$
$1-2/3=e^2$
$\Rightarrow e^2=1/3$ i.e. $e=\frac{1}{\sqrt3}$

thus length of Major axis $= \sqrt{2}$

minor axis $= \frac{2}{\sqrt{3}}$

eccentricity $= \frac{1}{\sqrt{3}}$

Find the condition that the St. line $x\cos\alpha + y\sin\alpha = p$ may be tangent to the ellipse

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$$

Solution – The equation of the ellipse

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$$

If (x_1, y_1) be the point of contact then equation of tangent at (x_1, y_1) is

$$\frac{xx_1}{a^2} + \frac{yy_1}{b^2} = 1 \quad \text{--- (I)}$$

If the St. line

$$x\cos\alpha + y\sin\alpha = p \quad \text{--- (II)}$$

is also tangent to the ellipse then (I) and (II) are same.

$$\therefore \frac{x_1}{a^2 \cdot \cos\alpha} = \frac{y_1}{b^2\sin\alpha} = \frac{1}{p}$$

$$\therefore x_1 = \frac{a^2\cos\alpha}{p} \Rightarrow x_1^2 = \frac{a^4\cos^2\alpha}{p^2}$$

$$y_1 = \frac{b^2\sin\alpha}{p} \Rightarrow y_1^2 = \frac{b^4\sin^2\alpha}{p^2}$$

Since (x_1, y_1) is point of contact

$$\Rightarrow \frac{x_1^2}{a^2} + \frac{y_1^2}{b^2} = 1$$

$$\frac{a^4\cos^2\alpha}{p^2 \cdot a^2} + \frac{b^4\sin^2\alpha}{p^2 \cdot b^2} = 1$$

$\Rightarrow a^2\cos^2\alpha + b^2\sin^2\alpha = p^2$ this is the required condition

3. Problem: Show that the equation of the locus of the ~~point~~ foot of the perpendicular from the centre of an ellipse on a tangent is

$$r^2 = a^2\cos^2\theta + b^2\sin^2\theta$$

a and b being the semi-major and semi-minor axes —

Solution → Let the equation of ellipse be

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$$

Any tangent to the ellipse is

$$y = mx + \sqrt{a^2m^2 + b^2} \quad \text{———} \quad (1)$$

m = slope of the tangent

Since slope of any line perpendicular to (1) $= -\frac{1}{m}$

∴ the equation of any line perpendicular to (1) and passing through C(0,0) is

$$\frac{y-0}{x-0} = -\frac{1}{m}$$

$$\Rightarrow m = -\frac{x}{y} \quad \text{———} \quad (11)$$

To get the locus of the foot of the perpendicular,

Eliminate m from (1) and (2)

Putting the value of m from (2) in (1), we get —

$$y = -\frac{x^2}{y} + \sqrt{a^2\frac{x^2}{y^2} + b^2}$$

$$x^2+y^2 = \sqrt{a^2\frac{x^2}{y^2}+b^2}$$

$$\Rightarrow (x^2+y^2)^2 = \frac{a^2x^2+b^2y^2}{~~p^2~~}$$

In order to get the locus in polar form put $x = r\cos\theta$, $y = r\sin\theta$ in the above equations

$\therefore x^2+y^2 = r^2$

$$r^4 = a^2r^2\cos^2\theta + b^2r^2\sin^2\theta$$

$$\Rightarrow r^4 = r^2(a^2\cos^2\theta + b^2\sin^2\theta)$$

$$\Rightarrow r^2 = a^2\cos^2\theta + b^2\sin^2\theta.$$

This is the required locus.

Problem: Prove that the product of perpendiculars from the foci on any tangent to an ellipse is equal to the square of the semi-minor axis, i.e. constant.

Solution – Let the equation of the ellipse be

$$\frac{x^2}{a^2}+\frac{y^2}{b^2}=1 \quad \text{——— (1)}$$

Where $b^2 = a^2(1-e^2)$

The equation of any tangent to the ellipse (1) is

$$y = mx + \sqrt{a^2m^2+b^2}$$

$$\Rightarrow mx - y + \sqrt{a^2m^2+b^2} = 0 \quad \text{——— (2)}$$

Since the foci of the ellipse (1) are $(ae, 0)$ and $(-ae, 0)$

Let p_1 and p_2 be the lengths of the perpendiculars drawn from $(ae, 0)$ and $(-ae, 0)$ to the tangents (8)

then $p_1 = \dfrac{mae + \sqrt{a^2m^2 + b^2}}{\sqrt{m^2 + (-1)^2}}$

$= \dfrac{mae + \sqrt{a^2m^2 + b^2}}{\sqrt{m^2+1}}$

$p_2 = \dfrac{-mae + \sqrt{a^2m^2 + b^2}}{\sqrt{m^2+1}}$

$\therefore p_1 \cdot p_2 = \dfrac{(a^2m^2 + b^2) - a^2m^2e^2}{m^2+1}$

$= \dfrac{a^2m^2(1-e^2) + b^2}{m^2+1}$

$= \dfrac{m^2b^2 + b^2}{m^2+1}$

$= \dfrac{b^2(m^2+1)}{m^2+1}$

$= b^2$

i.e the square of the semi-minor axis of the ellipse, which is constant.

15. Prove that the locus of the middle points of the portions of the tangents to an ellipse $\dfrac{x^2}{a^2} + \dfrac{y^2}{b^2} = 1$ included between the axes is the curve $\dfrac{a^2}{x^2} + \dfrac{b^2}{y^2} = 4$

Solution 2→

Since the Equation of Tangent to the Ellipse

$$\frac{x^2}{a^2}+\frac{y^2}{b^2}=1$$

at the point $P(a\cos\theta, b\sin\theta)$ is

$$\frac{x\cos\theta}{a}+\frac{y\sin\theta}{b}=1$$

or $$\frac{x/a}{\cos\theta}+\frac{y/b}{\sin\theta}=1$$

If this Tangent intersects the axes at A and B, then

$$OA=\frac{a}{\cos\theta} \qquad OB=\frac{b}{\sin\theta}$$

Therefore the Co-ordinate of A is $\left(\frac{a}{\cos\theta},0\right)$ and the Co-ordinate of B is $\left(0,\frac{b}{\sin\theta}\right)$

Let (x,y) be middle point of AB

then $$x=\frac{\frac{a}{\cos\theta}+0}{2} \qquad y=\frac{0+\frac{b}{\sin\theta}}{2}$$

$$\Rightarrow \cos\theta=\frac{a}{2x} \qquad \sin\theta=\frac{b}{2y}$$

Since $\cos^2\theta+\sin^2\theta=1$

$$\Rightarrow \frac{a^2}{4x^2}+\frac{b^2}{4y^2}=1$$

$$\Rightarrow \frac{a^2}{x^2}+\frac{b^2}{y^2}=4$$

This is the required Locus.